

Synchrony and synchronization

CN510 Guest lecture

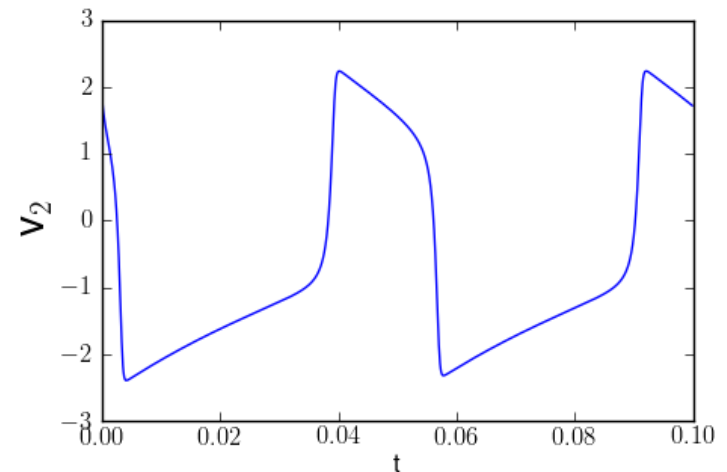
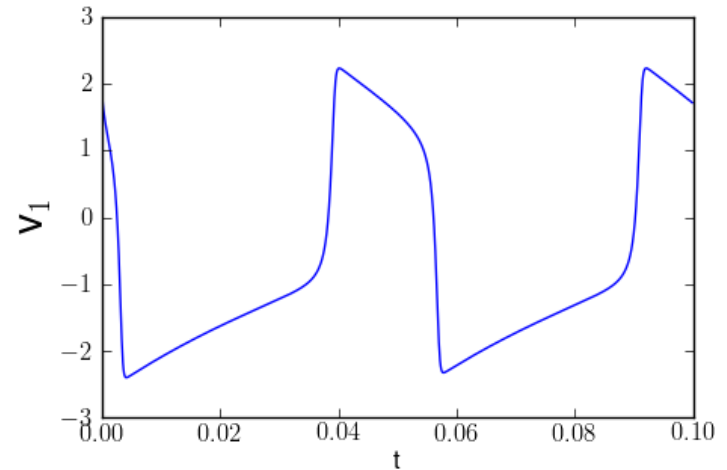
Robert Law

December 3, 2013

- Definitions and motivation
- Network connectivity and receptive fields / input trees
- The lattice of balanced equivalences
- Synchronization as subspace contraction

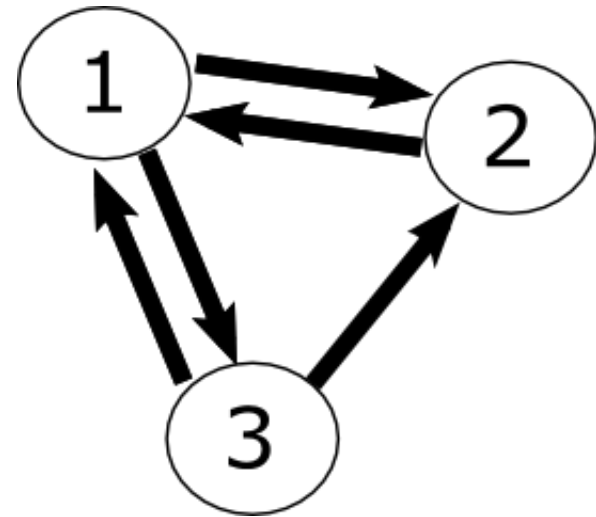
Synchrony

- Exact synchrony will refer to two model cells with exactly the same dynamics
- Here, $v_1(t) = v_2(t)$ for all t
- This is more precise than the synchrony neuroscientists typically refer to – but we can weaken the definition using an appropriate norm



Network structure and synchrony

- Suppose that every cell in this network were a Fitzhugh-Nagumo cell with identical coupling types, parameters, and initial conditions
- Which cells would remain synchronous with each other?

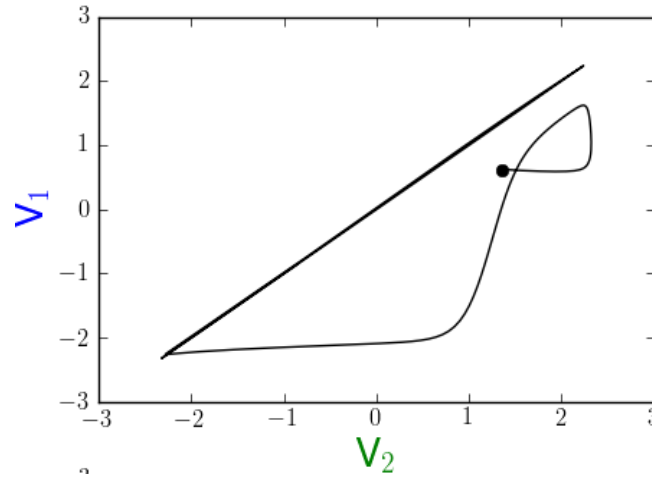


$$dv/dt = v - \frac{v^3}{3} - w + 0.5 + 0.2I$$

$$dw/dt = 0.08(v + 0.7 - 0.8w)$$

cf. Golubitsky, M. and Stewart, I. (2006). Nonlinear dynamics of networks: the groupoid formalism. *Bulletin of the American Mathematical Society*, 43(3):305.

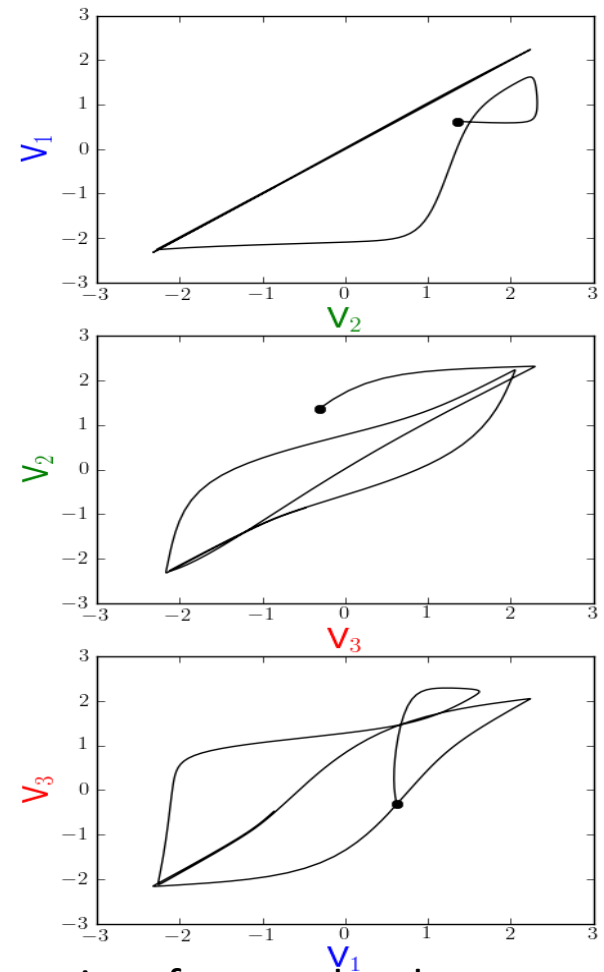
Synchrony and synchronization



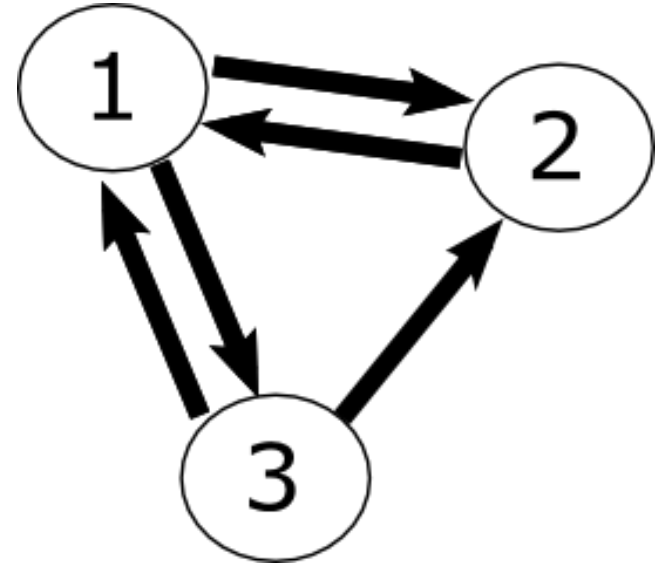
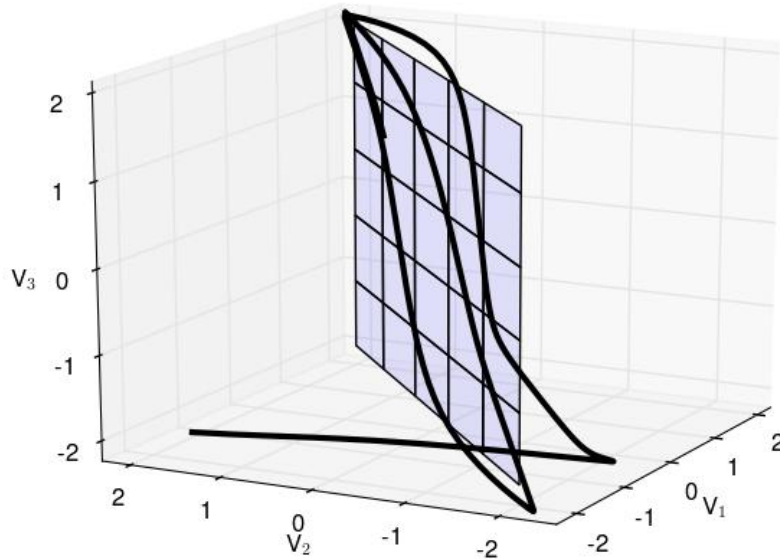
- So synchrony means trajectories are on the **diagonal** in phase space
- **Synchronization** of two cells is when their trajectories converge onto their respective diagonal

The polydiagonal

- Usually when we say synchrony we will be talking about “polysynchrony” or groups of synchronous cells
- This corresponds to “polydiagonals” which are generally of more than one dimension



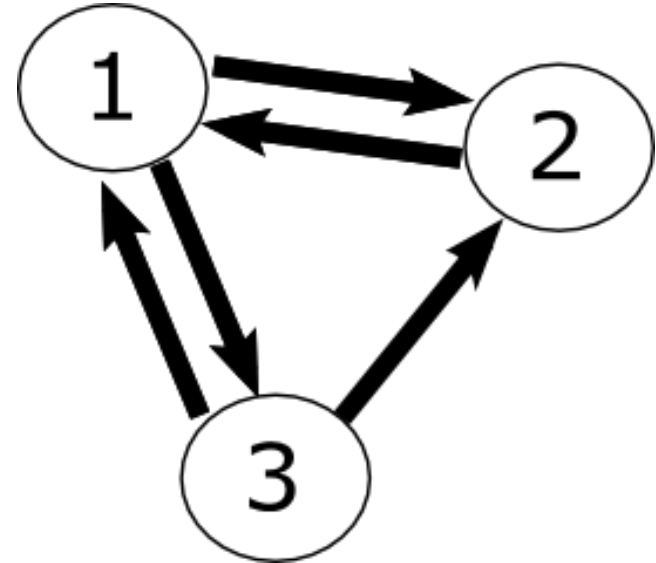
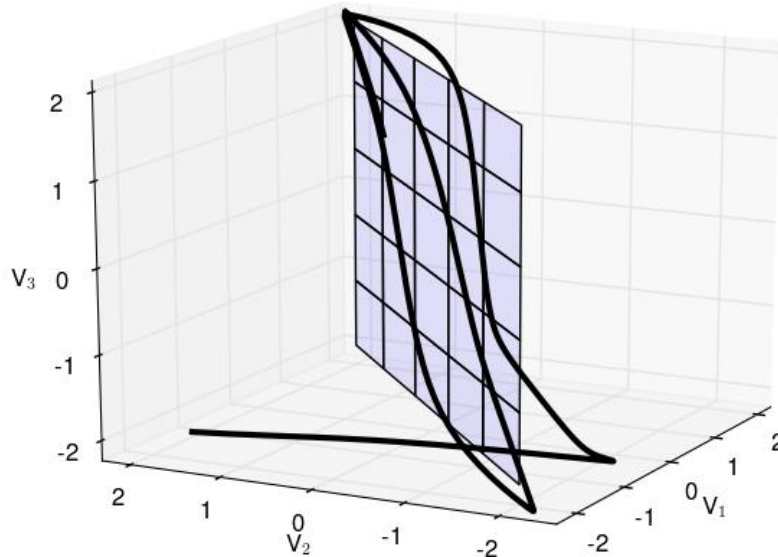
cf. Golubitsky, M. and Stewart, I. (2006). Nonlinear dynamics of networks: the groupoid formalism. *Bulletin of the American Mathematical Society*, 43(3):305.



- Here, we see synchronization onto the polydiagonal $V_1 = V_2 \neq V_3$

$$dv / dt = v - \frac{v^3}{3} - w + 0.5 + 0.2I$$

$$dw / dt = 0.08(v + 0.7 - 0.8w)$$



- What about this network's structure might lead us to believe that $v_1=v_2 \neq v_3$ is somehow special?

$$dv / dt = v - \frac{v^3}{3} - w + 0.5 + 0.2I$$

$$dw / dt = 0.08(v + 0.7 - 0.8w)$$

Receptive fields and synchrony

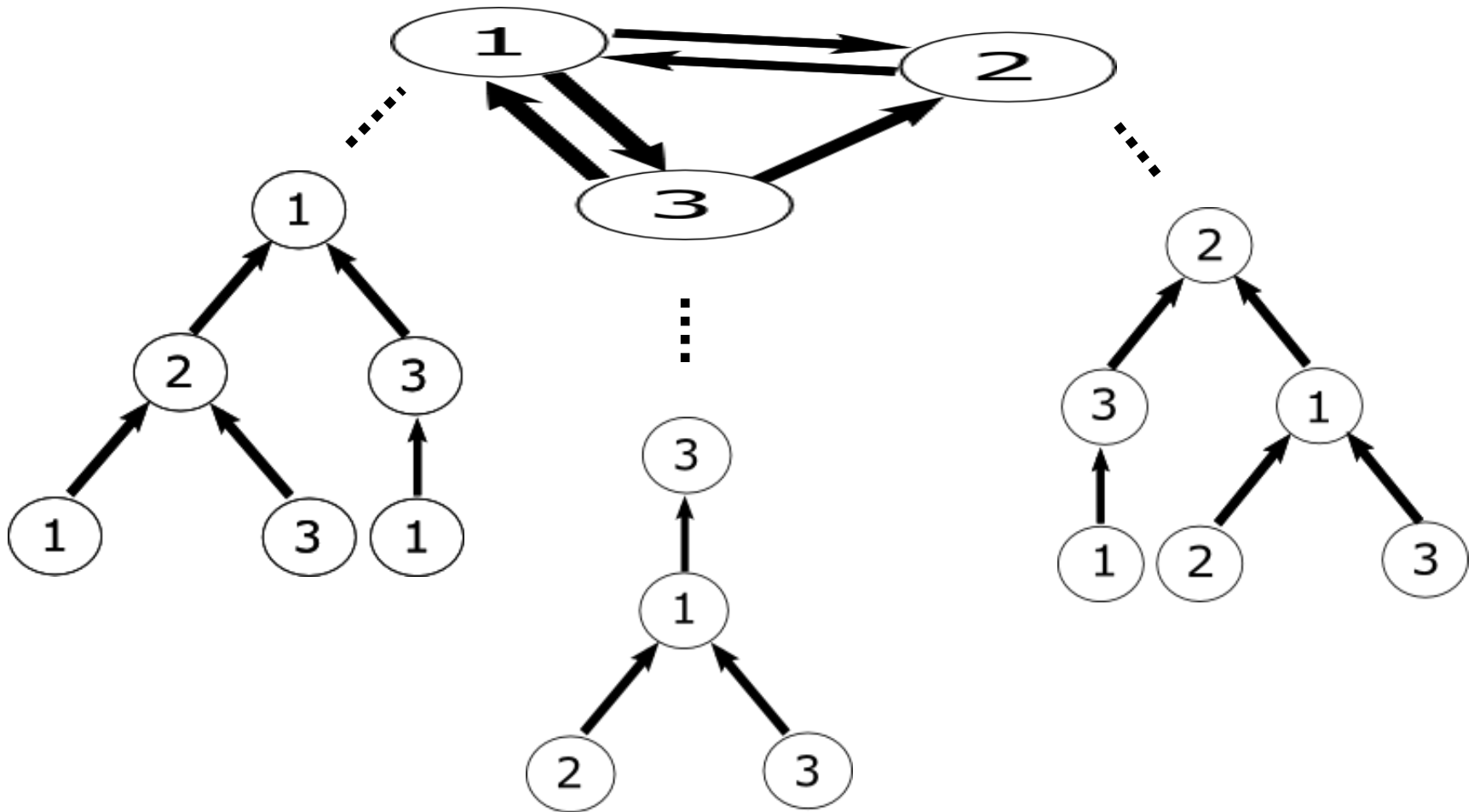
- Loosely, for two cells to have the same activity, either:
 - 1) they're very precisely tuned
 - or
 - 2) they're getting the same inputs

When do cells get the same inputs?

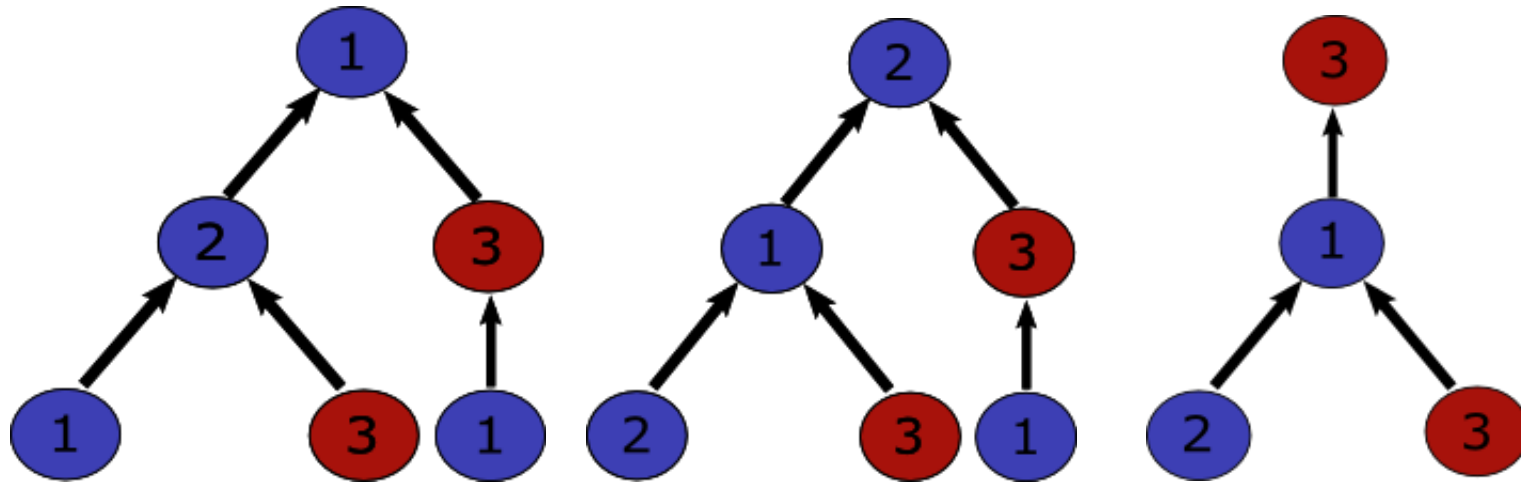
When their inputs get the same inputs

When their inputs get the same inputs

i.e. when their input trees “look the same”



Two types of trees

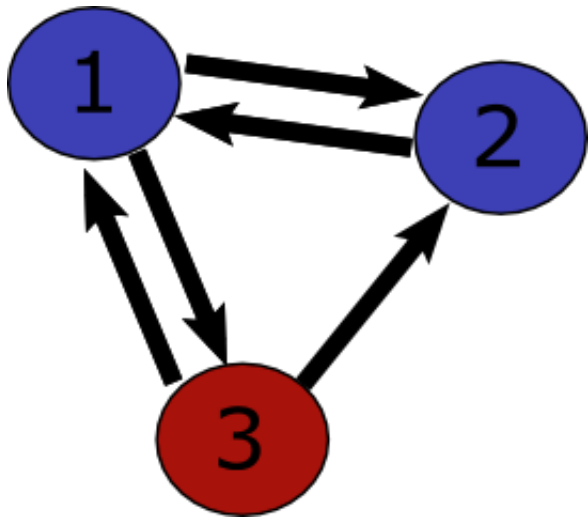


- Cells 1 and 2 get
 - 1 red input
 - 1 blue input
- Cell 3 gets
 - 1 blue input

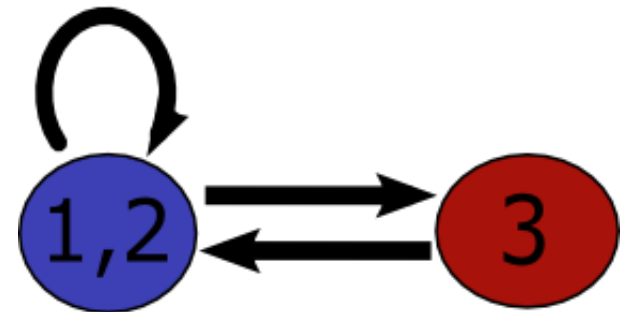
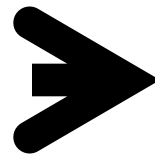
Trees of inputs

- If the input trees are the same shape (are **isomorphic**), cells can be synchronous
- We only need $n-1$ levels of the input trees to determine this (Norris's theorem)
- Computation in $n+m \log n$ where m is the number of edges and n is the number of nodes (Cardon/Crochemore partitioning algorithm)

Dimensionality reduction



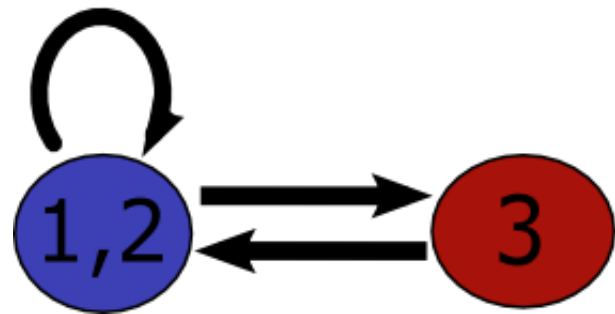
6 dimensions



4 dimensions

Preserve inputs while collapsing the graph

- Cells 1 and 2 get
 - 1 blue input
 - 1 red input
- Cell 3 gets
 - 1 blue input



Golubitsky, M. and Stewart, I. (2006). Nonlinear dynamics of networks: the groupoid formalism. *Bulletin of the American Mathematical Society*, 43(3):305.

Boldi, P. and Vigna, S. (2002). Fibrations of graphs. *Discrete Mathematics*, 243(1-3):21{66.

Polydiagonals and receptive fields

- The colors here represent synchronous groups of cells
- There's more to this!
- A network's nodes may be partitioned (colored) in many different ways – the colorations that are internally consistent are called “balanced”
- All the balanced colorations are “legal” ways for synchrony to occur in a network

What are the balanced colorations for this network?

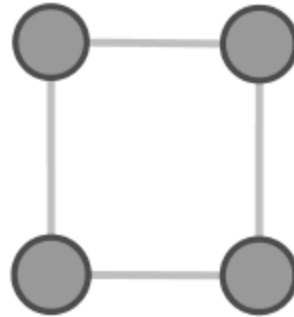


Figure 2.6: Four-cell network C_4 . All connections are bidirectional.

Balanced partitions form a complete lattice

Bottom is total synchrony,
top is no synchrony

Upward segments indicate
partition refinement

Lattice theory is a topic for
another lecture, but this
ordering has implications
for synchronization

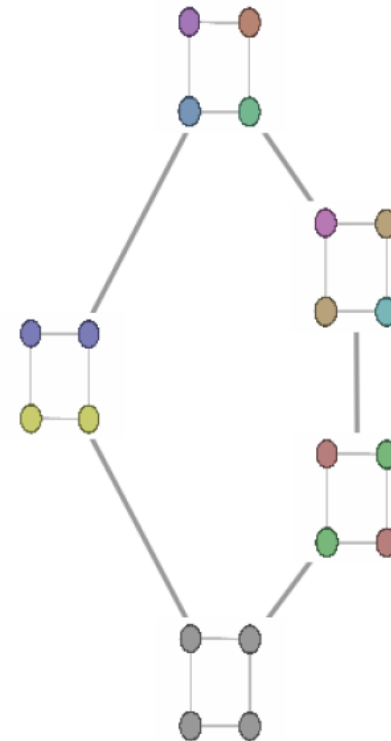


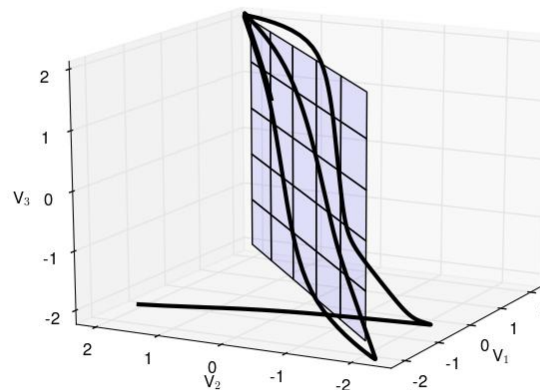
Figure 2-7: Hasse diagram for the balanced equivalence lattice for C_4 .

Synchronization

- We have now associated a system of linear subspaces with synchrony
- Addressing synchronization amounts to the question: “Are these subspaces attractor manifolds?”
- Up until now, we have depended only on the graph structure, but to determine this will depend on the geometry of the system given by its Jacobian matrix

Contraction

- We will look at the Jacobian projected into a subspace orthogonal to a chosen polydiagonal subspace – we will call such a space a “transdiagonal” subspace – and show that under certain conditions, this subspace **contracts**



Contraction

Given a system of ordinary differential equations, we have the linearization

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} = \mathbf{J} \delta \mathbf{x}$$

And its transpose

$$\delta\dot{\mathbf{x}}^T = \delta \mathbf{x}^T \frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}$$

describing how a virtual displacement affects a virtual velocity. In other words, if we moved a trajectory to a nearby trajectory, how much would the velocity change?

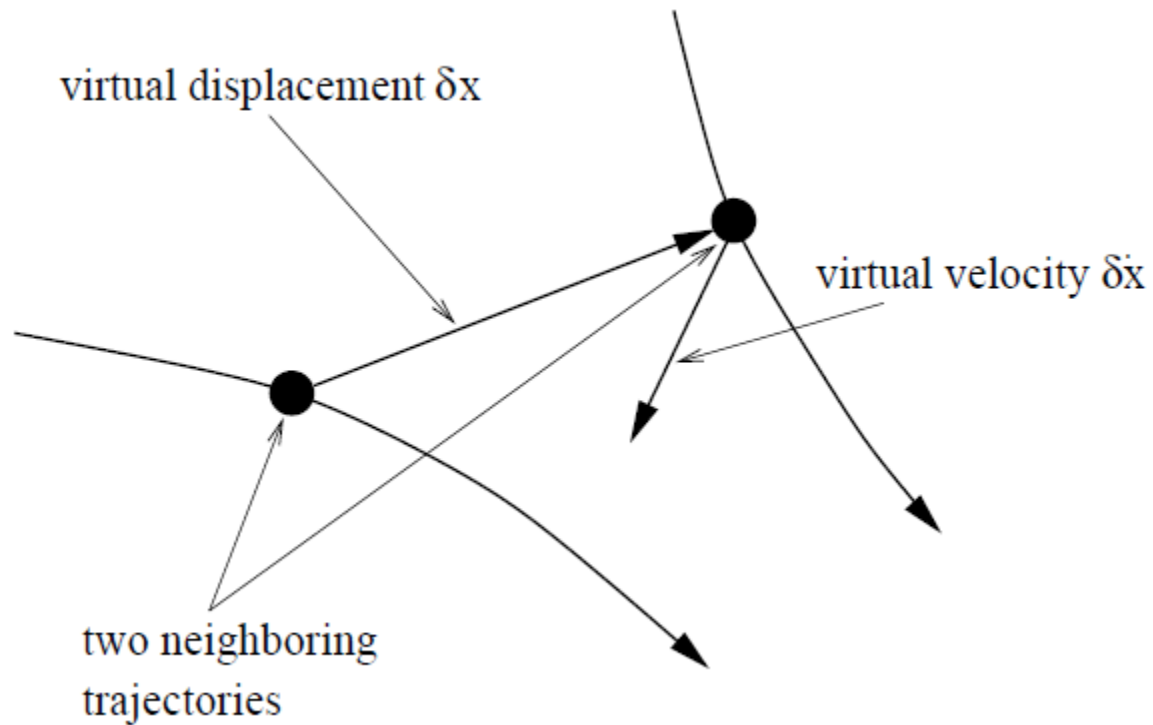


Figure 1 from Lohmiller, W., & Slotine, J. J. E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6), 683-696

Contraction

Multiply together to get a squared distance and take the time derivative:

$$\frac{d}{dt}(\delta\mathbf{x}^T \delta\mathbf{x}) = \delta\mathbf{x}^T \delta\dot{\mathbf{x}} + \delta\dot{\mathbf{x}}^T \delta\mathbf{x} = \delta\mathbf{x}^T \mathbf{J} \delta\mathbf{x} + \delta\mathbf{x}^T \mathbf{J}^T \delta\mathbf{x}$$

Combine terms to get

$$\frac{d}{dt}(\delta\mathbf{x}^T \delta\mathbf{x}) = 2\delta\mathbf{x}^T \mathbf{J}_s \delta\mathbf{x}$$

where $\mathbf{J}_s = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T)$ is the symmetric part of the Jacobian matrix.

Since the system is locally linear, at each point we have

$$\frac{d}{dt}(\delta\mathbf{x}^T \delta\mathbf{x}) \leq 2 \lambda_{max} \delta\mathbf{x}^T \delta\mathbf{x}$$

Viewed as a linear ODE, the largest eigenvalue bounds the solution

If the largest eigenvalue is negative, i.e. the Jacobian is **negative definite**, then the distance between any two trajectories goes to zero

- This is a generalization of analyzing the Jacobian at a fixed point – in that case we looked for all eigenvalues having negative real part at that point
- If an equilibrium exists, all trajectories converge onto it - i.e. the entire space **contracts** to a point
- It can be shown that all autonomous systems have at least one
- We can then look at **subspaces** and see if they contract

Norm considerations

- Eigenvalues are defined given vectors with a 2-norm
- **But because synchrony is norm-invariant, we can use any norm**
- The matrix 1-norm and ∞ -norm seem to usually be the most convenient ones

Example

- Two-cell Fitzhugh-Nagumo system – we will fix all parameters except a connection weight c
- We can compute parameter ranges that will lead either to synchrony or antisynchrony (where $x_1 = -x_2$)

Example 2.3.1.

Consider two identical bidirectionally coupled Fitzhugh-Nagumo model cells (Fitzhugh, 1961; Nagumo et al., 1962) with identical linear connection weights c :

$$\dot{v}_i = v_i - \frac{v_i^3}{3} - u_i + cv_j \quad (2.12)$$

$$\dot{u}_i = (v_i - a - bu_i)\omega \quad (2.13)$$

for $a, b, \omega > 0$. The Jacobian for the system is

$$\mathbf{J} = \begin{bmatrix} 1 - v_1^2 & \omega & c & 0 \\ -1 & -b\omega & 0 & 0 \\ c & 0 & 1 - v_2^2 & \omega \\ 0 & 0 & -1 & -b\omega \end{bmatrix} \quad (2.14)$$

The symmetric part is:

$$\mathbf{J}_s = \begin{bmatrix} 1 - v_1^2 & \frac{1}{2}(\omega - 1) & c & 0 \\ \frac{1}{2}(\omega - 1) & -b\omega & 0 & 0 \\ c & 0 & 1 - v_2^2 & \frac{1}{2}(\omega - 1) \\ 0 & 0 & \frac{1}{2}(\omega - 1) & -b\omega \end{bmatrix} \quad (2.15)$$

1. *Synchronization* is contraction to the polydiagonal subspace:

$$\mathbf{B}_{min} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Its nullspace is:

$$\mathbf{B}_{\perp} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Changing basis into this space yields:

$$\mathbf{J}_{\perp} = \begin{pmatrix} -2.0c - 1.0v_0^2 - 1.0v_1^2 + 2.0 & 1.0\omega - 1.0 \\ 1.0\omega - 1.0 & -2.0b\omega \end{pmatrix}$$

leading to constraints

$$\begin{aligned} -2c + |\omega - 1| + 2 &< 0 \\ -2b\omega + \omega - 1 &< 0 \end{aligned}$$

We will reuse Eqs. 2.12 and 2.13, supposing¹¹ that $\omega = 1/13$ and $b = 8/10$. Then the second inequality in 2.16 holds everywhere and the first is sensitive only to c . In particular, $c < -29/20$ guarantees antisynchronization and $c > 29/20$ guarantees synchronization. Initial conditions were set to a random value from the uniform distribution $[-0.1, 0.1]$.

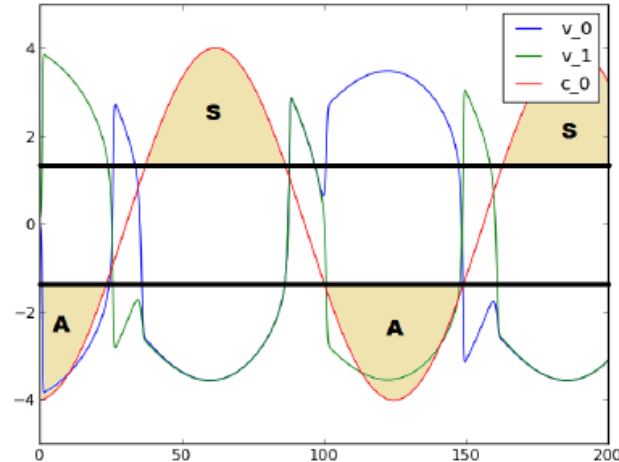
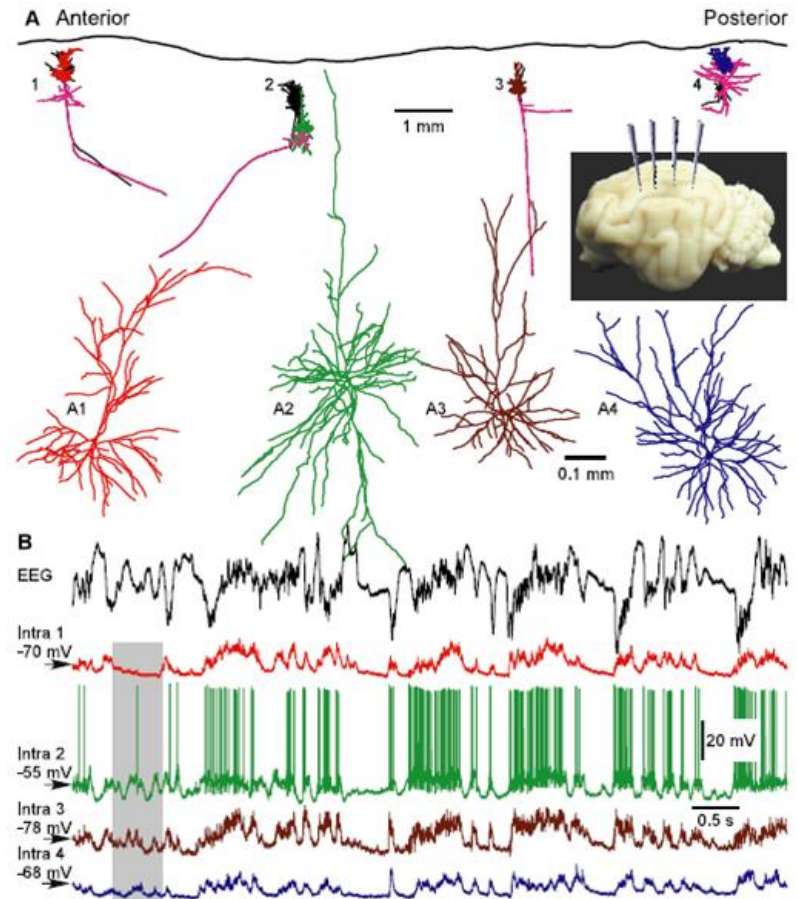


Figure 2-20: Two-cell model system with voltages v_0, v_1 with varying global weighting parameter c (recovery variable u not shown). Regions marked A indicate the antisynchronization regime while regions marked S mark the synchronization regime.

But synchrony in the brain?

- While exact synchrony is a theoretical construct, waveforms with strong temporal correlations do occur in living brains
- So how should we relax our definitions?



Volgushev, M., Chauvette, S., Mukovski, M., and Timofeev, I. (2006). Precise long range synchronization of activity and silence in neocortical neurons during slow wave sleep. *Journal of neuroscience*, 26(21):5665-5672.

(later corrected to indicate ketamine-xylazine anaesthesia)

Remarks on connectivity

- The number of balanced colorations in all-to-all networks becomes huge as n increases
 - $P(n)$ grows roughly as $\exp(n^{1/2})$
- But all-to-all connectivity is not “how the brain works”
 - On the order of 10^{10} neurons and 10^{14} synapses
- However, *in-regular* connectivity guarantees that all cells have identical input trees, leading to many balanced colorations

References

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